**TABLE 6.3**
## Continuous distributions

| **Uniform** | **U($a$,$b$)** |
|---|---|
| Possible applications | Used as a "first" model for a quantity that is felt to be randomly varying between $a$ and $b$ but about which little else is known. The U(0,1) distribution is essential in generating random values from all other distributions (see Chaps. 7 and 8) |
| Density (see Fig. 6.1) | $f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$ |
| Distribution | $F(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b < x \end{cases}$ |
| Parameters | $a$ and $b$ real numbers with $a < b$; $a$ is a location parameter, $b - a$ is a scale parameter |
| Range | $[a,b]$ |
| Mean | $\dfrac{a+b}{2}$ |
| Variance | $\dfrac{(b-a)^2}{12}$ |
| Mode | Does not uniquely exist |
| MLE | $\hat{a} = \min_{1\le i\le n} X_i$, $\hat{b} = \max_{1\le i\le n} X_i$ |
| Comments | 1. The U(0,1) distribution is a special case of the beta distribution (when $\alpha_1 = \alpha_2 = 1$)<br>2. If $X \sim$ U(0,1) and $[x, x+\Delta x]$ is a subinterval of [0,1] with $\Delta x \ge 0$, $$P(X \in [x, x+\Delta x]) = \int_x^{x+\Delta x} 1\,dy = (x+\Delta x) - x = \Delta x$$ which justifies the name "uniform" |

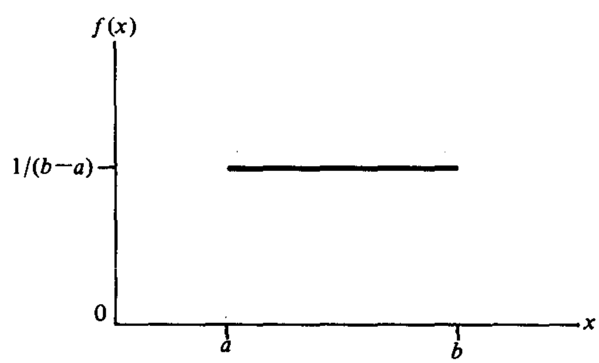


**FIGURE 6.1**
U($a$,$b$) density function.

| **Exponential** | **expo($\beta$)** |
|---|---|
| Possible applications | Interarrival times of "customers" to a system that occur at a constant rate |
| Density (see Fig. 6.2) | $f(x) = \begin{cases} \dfrac{1}{\beta} e^{-x/\beta} & \text{if } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$ |

TABLE 6.3 (*continued*)

| **Exponential** | **expo($\beta$)** |
|---|---|
| Distribution | $F(x) = \begin{cases} 1 - e^{-x/\beta} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$ |
| Parameter | Scale parameter $\beta > 0$ |
| Range | $[0,\infty)$ |
| Mean | $\beta$ |
| Variance | $\beta^2$ |
| Mode | 0 |
| MLE | $\hat{\beta} = \bar{X}(n)$ |
| Comments | 1. The expo($\beta$) distribution is a special case of both the gamma and Weibull distributions (for shape parameter $\alpha = 1$ and scale parameter $\beta$ in both cases)<br>2. If $X_1, X_2, \ldots, X_m$ are independent expo($\beta$) random variables, then $X_1 + X_2 + \cdots + X_m \sim$ gamma($m,\beta$), also called the *m-Erlang distribution*<br>3. The exponential distribution is the only continuous distribution with the memoryless property (see Prob. 4.26) |

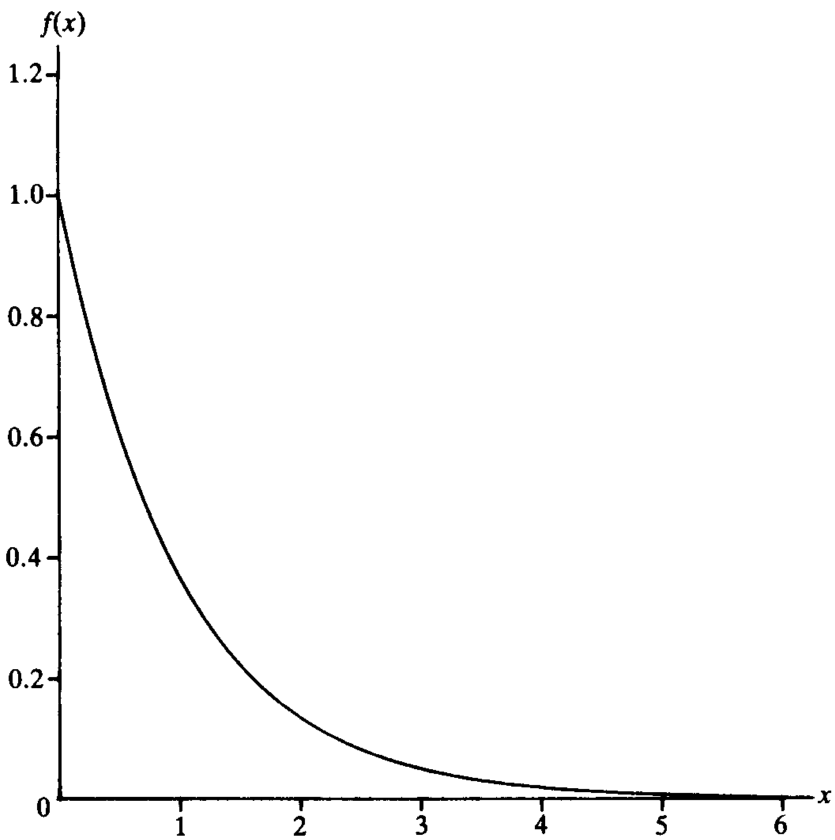


**FIGURE 6.2**
expo(1) density function.

| **Gamma** | **gamma($\alpha,\beta$)** |
|---|---|
| Possible applications | Time to complete some task, e.g., customer service or machine repair |

**TABLE 6.3** *(continued)*

| Gamma | gamma($\alpha,\beta$) |
|---|---|
| Density (see Fig. 6.3) | $f(x) = \begin{cases} \dfrac{\beta^{-\alpha}x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)} & \text{if } x>0 \\ 0 & \text{otherwise} \end{cases}$ <br> where $\Gamma(\alpha)$ is the *gamma function*, defined by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ for any real number $z>0$. Some properties of the gamma function: $\Gamma(z+1) = z\Gamma(z)$ for any $z>0$, $\Gamma(k+1) = k!$ for any nonnegative integer $k$, $\Gamma(k+\frac{1}{2}) = \sqrt{\pi}\cdot 1\cdot 3\cdot 5\cdots(2k-1)/2^k$ for any positive integer $k$, $\Gamma(1/2) = \sqrt{\pi}$ |
| Distribution | If $\alpha$ is not an integer, there is no closed form. If $\alpha$ is a positive integer, then <br> $F(x) = \begin{cases} 1 - e^{-x/\beta}\displaystyle\sum_{j=0}^{\alpha-1}\frac{(x/\beta)^j}{j!} & \text{if } x>0 \\ 0 & \text{otherwise} \end{cases}$ |
| Parameters | Shape parameter $\alpha>0$, scale parameter $\beta>0$ |
| Range | $[0,\infty)$ |
| Mean | $\alpha\beta$ |
| Variance | $\alpha\beta^2$ |
| Mode | $\beta(\alpha-1)$ if $\alpha \geq 1$, 0 if $\alpha<1$ |
| MLE | The following two equations must be satisfied: <br> $\ln\hat{\beta} + \Psi(\hat{\alpha}) = \dfrac{\sum_{i=1}^{n}\ln X_i}{n}, \qquad \hat{\alpha}\hat{\beta} = \bar{X}(n)$ <br> which could be solved numerically. [$\Psi(\hat{\alpha}) = \Gamma'(\hat{\alpha})/\Gamma(\hat{\alpha})$ and is called the *digamma function*; $\Gamma'$ denotes the derivative of $\Gamma$.] Alternatively, approximations to $\hat{\alpha}$ and $\hat{\beta}$ can be obtained by letting $T = [\ln\bar{X}(n) - \Sigma_{i=1}^{n}\ln X_i/n]^{-1}$, using Table 6.19 (see App. 6A) to obtain $\hat{\alpha}$ as a function of $T$, and letting $\hat{\beta} = \bar{X}(n)/\hat{\alpha}$. [See Choi and Wette (1969) for the derivation of this procedure and of Table 6.19] |
| Comments | 1. The expo($\beta$) and gamma($1,\beta$) distributions are the same <br> 2. For a positive integer $m$, the gamma($m,\beta$) distribution is called the $m$-Erlang($\beta$) distribution <br> 3. The chi-square distribution with $k$ df is the same as the gamma($k/2, 2$) distribution <br> 4. If $X_1, X_2, \ldots, X_m$ are independent random variables with $X_i \sim$ gamma($\alpha_i,\beta$), then $X_1 + X_2 + \cdots + X_m \sim$ gamma($\alpha_1 + \alpha_2 + \cdots + \alpha_m, \beta$) <br> 5. If $X_1$ and $X_2$ are independent random variables with $X_i \sim$ gamma($\alpha_i,\beta$), then $X_1/(X_1+X_2) \sim$ beta($\alpha_1,\alpha_2$) <br> 6. $X \sim$ gamma($\alpha,\beta$) if and only if $Y = 1/X$ has a Pearson type V distribution with shape and scale parameters $\alpha$ and $1/\beta$, denoted PT5($\alpha,1/\beta$) <br> 7. $\lim_{x\to 0} f(x) = \begin{cases} \infty & \text{if } \alpha<1 \\ \dfrac{1}{\beta} & \text{if } \alpha=1 \\ 0 & \text{if } \alpha>1 \end{cases}$ |

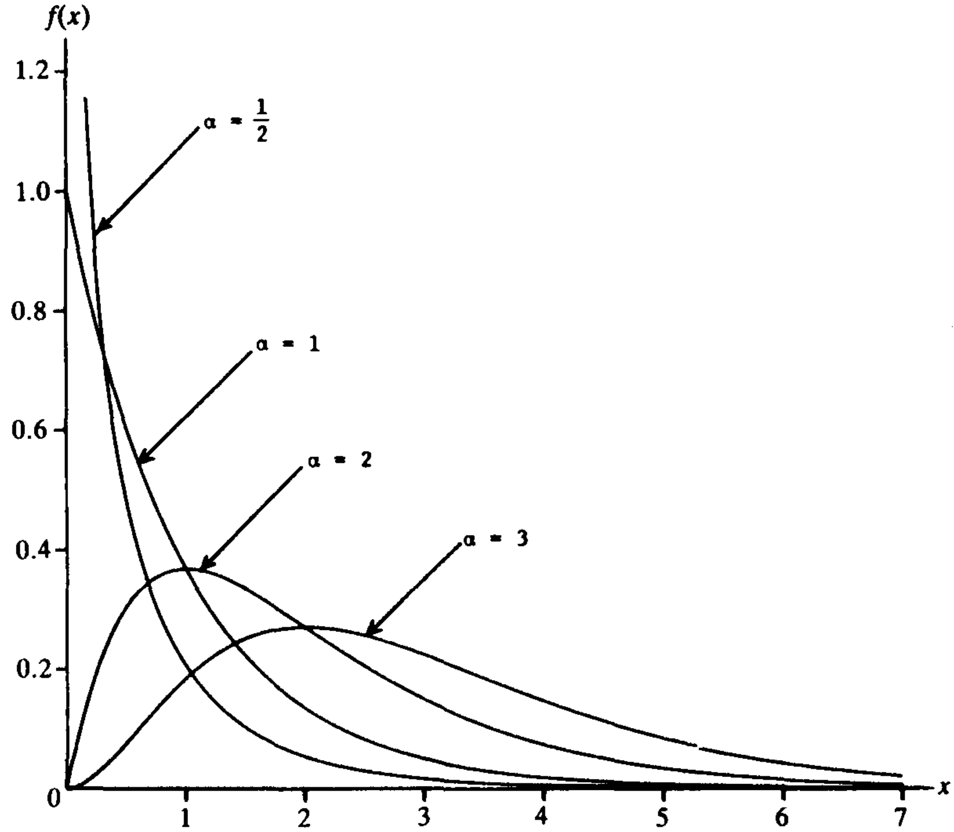


**FIGURE 6.3**
gamma($\alpha$,1) density functions.

**TABLE 6.3** (*continued*)

| **Weibull** | **Weibull($\alpha, \beta$)** |
|---|---|
| Possible applications | Time to complete some task (density takes on shapes similar to gamma densities), time to failure of a piece of equipment |
| Density (see Fig. 6.4) | $f(x) = \begin{cases} \alpha\beta^{-\alpha}x^{\alpha-1}e^{-(x/\beta)^\alpha} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$ |
| Distribution | $F(x) = \begin{cases} 1 - e^{-(x/\beta)^\alpha} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$ |
| Parameters | Shape parameter $\alpha > 0$, scale parameter $\beta > 0$ |
| Range | $[0,\infty)$ |
| Mean | $\frac{\beta}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)$ |
| Variance | $\frac{\beta^2}{\alpha}\left\{2\Gamma\left(\frac{2}{\alpha}\right) - \frac{1}{\alpha}\left[\Gamma\left(\frac{1}{\alpha}\right)\right]^2\right\}$ |
| Mode | $\begin{cases} \beta\left(\frac{\alpha-1}{\alpha}\right)^{1/\alpha} & \text{if } \alpha \geq 1 \\ 0 & \text{if } \alpha < 1 \end{cases}$ |

**TABLE 6.3** *(continued)*

| **Weibull** | **Weibull($\alpha,\beta$)** |
|---|---|
| MLE | The following two equations must be satisfied: $$\frac{\sum_{i=1}^{n} X_i^{\hat{\alpha}} \ln X_i}{\sum_{i=1}^{n} X_i^{\hat{\alpha}}} - \frac{1}{\hat{\alpha}} = \frac{\sum_{i=1}^{n} \ln X_i}{n}, \qquad \hat{\beta} = \left(\frac{\sum_{i=1}^{n} X_i^{\hat{\alpha}}}{n}\right)^{1/\hat{\alpha}}$$ The first can be solved for $\hat{\alpha}$ numerically by Newton's method, and the second equation then gives $\hat{\beta}$ directly. The general recursive step for the Newton iterations is $$\hat{\alpha}_{k+1} = \hat{\alpha}_k + \frac{A + 1/\hat{\alpha}_k - C_k/B_k}{1/\hat{\alpha}_k^2 + (B_k H_k - C_k^2)/B_k^2}$$ where $$A = \frac{\sum_{i=1}^{n} \ln X_i}{n}, \qquad B_k = \sum_{i=1}^{n} X_i^{\hat{\alpha}_k}, \qquad C_k = \sum_{i=1}^{n} X_i^{\hat{\alpha}_k} \ln X_i$$ and $$H_k = \sum_{i=1}^{n} X_i^{\hat{\alpha}_k} (\ln X_i)^2$$ [See Thoman, Bain, and Antle (1969) for these formulas, as well as for confidence intervals on the true $\alpha$ and $\beta$.] As a starting point for the iterations, the estimate $$\hat{\alpha}_0 = \left\{\frac{\frac{6}{\pi^2}\left[\sum_{i=1}^{n} (\ln X_i)^2 - \left(\sum_{i=1}^{n} \ln X_i\right)^2 \Big/ n\right]}{n-1}\right\}^{-1/2}$$ [due to Menon (1963) and suggested in Thoman, Bain, and Antle (1969)] may be used. With this choice of $\hat{\alpha}_0$, it was reported in Thoman, Bain, and Antle (1969) that an average of only 3.5 Newton iterations were needed to achieve four-place accuracy. |
| Comments | 1. The expo($\beta$) and Weibull($1,\beta$) distributions are the same<br>2. $X \sim$ Weibull($\alpha,\beta$) if and only if $X^\alpha \sim$ expo($\beta^\alpha$) (see Prob. 6.2)<br>3. The (natural) logarithm of a Weibull random variable has a distribution known as the *extreme-value* or *Gumbel distribution* [see Law and Vincent (1990), Lawless (1982), and Prob. 8.1(*b*)]<br>4. The Weibull($2, \beta$) distribution is also called a *Rayleigh distribution* with parameter $\beta$, denoted Rayleigh($\beta$). If $Y$ and $Z$ are independent normal random variables with mean 0 and variance $\beta^2$ (see the normal distribution), then $X = (Y^2 + Z^2)^{1/2} \sim$ Rayleigh($2^{1/2}\beta$)<br>5. As $\alpha \to \infty$, the Weibull distribution becomes degenerate at $\beta$. Thus, Weibull densities for large $\alpha$ have a sharp peak at the mode<br>6. $$\lim_{x \to 0} f(x) = \begin{cases} \infty & \text{if } \alpha < 1 \\ \frac{1}{\beta} & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1 \end{cases}$$ |

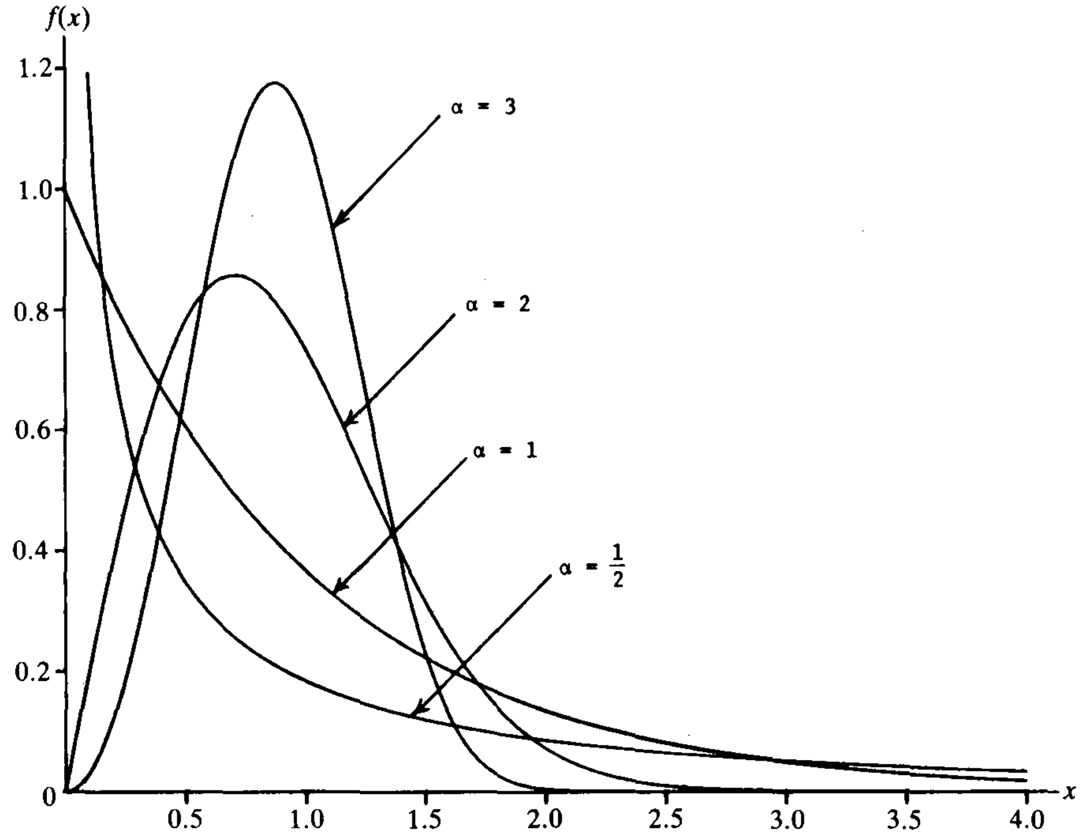


**FIGURE 6.4**
Weibull($\alpha$,1) density functions.

**TABLE 6.3** *(continued)*

| **Normal** | **N($\mu,\sigma^2$)** |
|---|---|
| Possible applications | Errors of various types, e.g., in the impact point of a bomb; quantities that are the sum of a large number of other quantities (by virtue of central limit theorems) |
| Density (see Fig. 6.5) | $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$ for all real numbers $x$ |
| Distribution | No closed form |
| Parameters | Location parameter $\mu \in (-\infty,\infty)$, scale parameter $\sigma > 0$ |
| Range | $(-\infty,\infty)$ |
| Mean | $\mu$ |
| Variance | $\sigma^2$ |
| Mode | $\mu$ |
| MLE | $\hat{\mu} = \bar{X}(n)$, $\hat{\sigma} = \left[\frac{n-1}{n} S^2(n)\right]^{1/2}$ |
| Comments | 1. If two jointly distributed normal random variables are uncorrelated, they are also independent. For distributions other than normal, this implication is not true in general |

**TABLE 6.3** *(continued)*

| **Normal** | **N($\mu$,$\sigma^2$)** |
|---|---|
| | 2. Suppose that the joint distribution of $X_1, X_2, \ldots, X_m$ is multivariate normal and let $\mu_i = E(X_i)$ and $C_{ij} = \mathrm{Cov}(X_i, X_j)$. Then for any real numbers $a, b_1, b_2, \ldots, b_m$, the random variable $a + b_1X_1 + b_2X_2 + \cdots + b_mX_m$ has a normal distribution with mean $\mu = a + \sum_{i=1}^m b_i\mu_i$ and variance $$\sigma^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} b_i b_j C_{ij}$$ Note that we need *not* assume independence of the $X_i$'s. If the $X_i$'s *are* independent, then $$\sigma^2 = \sum_{i=1}^{m} b_i^2 \mathrm{Var}(X_i)$$ |
| | 3. The N(0,1) distribution is often called the *standard* or *unit normal distribution* |
| | 4. If $X_1, X_2, \ldots, X_k$ are independent standard normal random variables, then $X_1^2 + X_2^2 + \cdots + X_k^2$ has a chi-square distribution with $k$ df, which is also the gamma($k/2$,2) distribution |
| | 5. If $X \sim \mathrm{N}(\mu,\sigma^2)$, then $e^X$ has the *lognormal distribution* with parameters $\mu$ and $\sigma$, denoted LN($\mu$,$\sigma^2$) |
| | 6. If $X \sim \mathrm{N}(0,1)$, if $Y$ has a chi-square distribution with $k$ df, and if $X$ and $Y$ are independent, then $X/\sqrt{Y/k}$ has a $t$ distribution with $k$ df (sometimes called *Student's t distribution*) |
| | 7. If the normal distribution is used to represent a nonnegative quantity (e.g., time), then its density should be truncated at $x = 0$ (see Sec. 6.8) |
| | 8. As $\sigma \to 0$, the normal distribution becomes degenerate at $\mu$ |

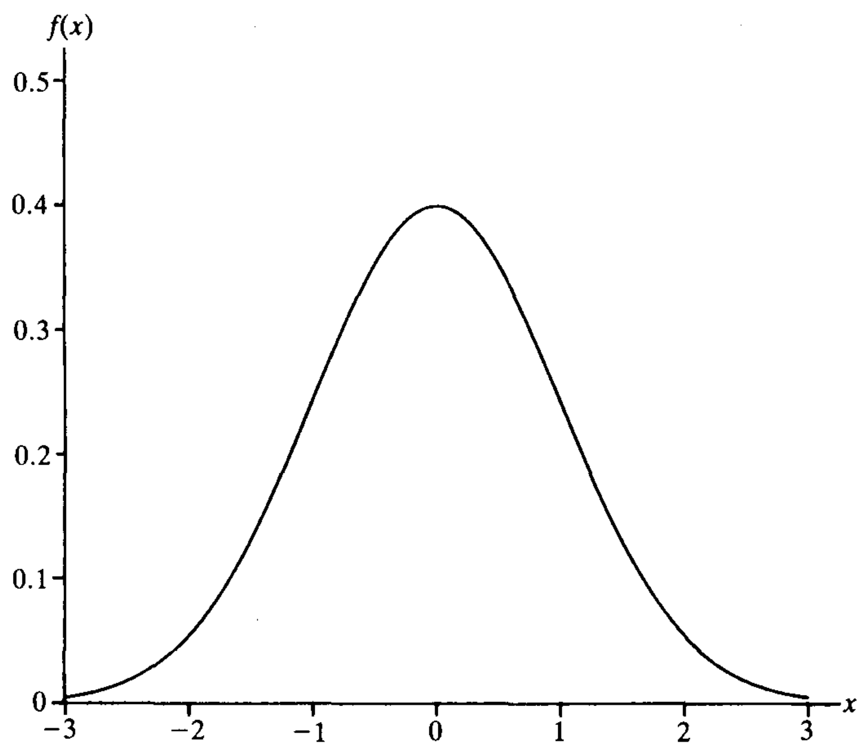


**FIGURE 6.5**
N(0,1) density function.

**TABLE 6.3** (*continued*)

| Lognormal | $\mathrm{LN}(\mu,\sigma^2)$ |
|---|---|
| Possible applications | Time to perform some task [density takes on shapes similar to gamma$(\alpha,\beta)$ and Weibull$(\alpha,\beta)$ densities for $\alpha>1$, but can have a large "spike" close to $x=0$ that is often useful]; quantities that are the product of a large number of other quantities (by virtue of central limit theorems) |
| Density (see Fig. 6.6) | $f(x)=\begin{cases}\dfrac{1}{x\sqrt{2\pi\sigma^2}}\exp\dfrac{-(\ln x-\mu)^2}{2\sigma^2} & \text{if } x>0\\ 0 & \text{otherwise}\end{cases}$ |
| Distribution | No closed form |
| Parameters | Shape parameter $\sigma>0$, scale parameter $\mu\in(-\infty,\infty)$ |
| Range | $[0,\infty)$ |
| Mean | $e^{\mu+\sigma^2/2}$ |
| Variance | $e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$ |
| Mode | $e^{\mu-\sigma^2}$ |
| MLE | $\hat{\mu}=\dfrac{\sum_{i=1}^{n}\ln X_i}{n},\quad \hat{\sigma}=\left[\dfrac{\sum_{i=1}^{n}(\ln X_i-\hat{\mu})^2}{n}\right]^{1/2}$ |
| Comments | 1. $X\sim\mathrm{LN}(\mu,\sigma^2)$ if and only if $\ln X\sim\mathrm{N}(\mu,\sigma^2)$. Thus, if one has data $X_1, X_2, \ldots, X_n$ that are thought to be lognormal, the logarithms of the data points, $\ln X_1, \ln X_2, \ldots, \ln X_n$, can be treated as normally distributed data for purposes of hypothesizing a distribution, parameter estimation, and goodness-of-fit testing<br>2. As $\sigma\to 0$, the lognormal distribution becomes degenerate at $e^{\mu}$. Thus, lognormal densities for small $\sigma$ have a sharp peak at the mode<br>3. $\lim_{x\to 0} f(x)=0$, regardless of the parameter values |

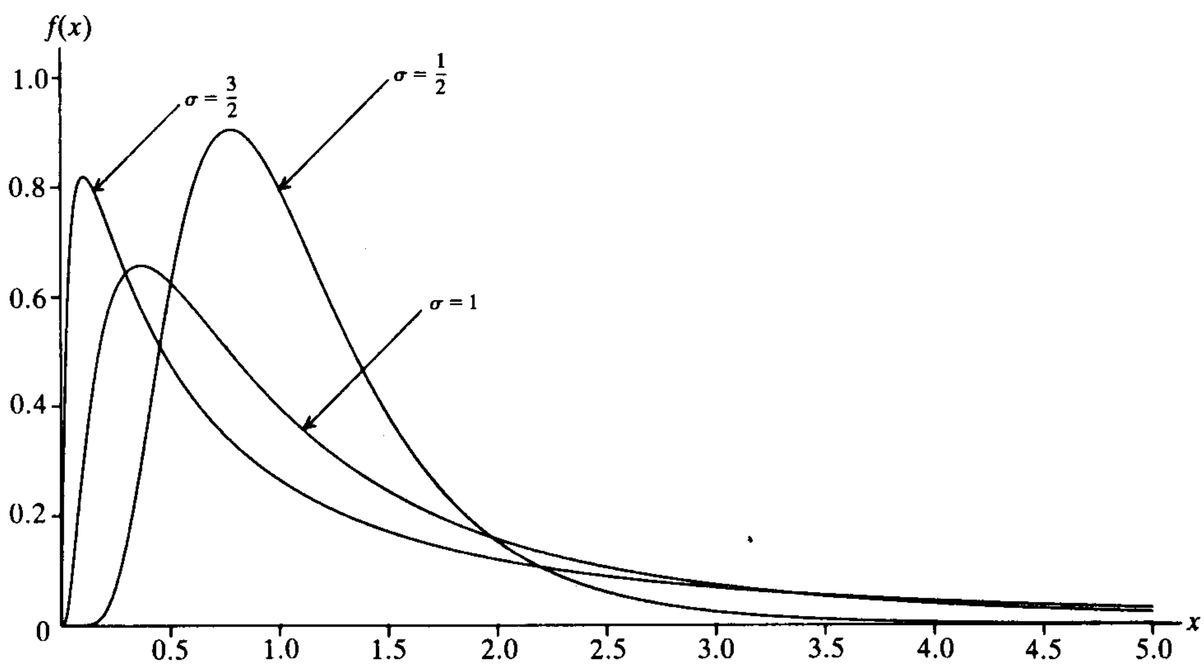


**FIGURE 6.6**
$\mathrm{LN}(0,\sigma^2)$ density functions.

**TABLE 6.3** *(continued)*

| Beta | beta($\alpha_1,\alpha_2$) |
|---|---|
| Possible applications | Used as a rough model in the absence of data (see Sec. 6.9); distribution of a random proportion, such as the proportion of defective items in a shipment; time to complete a task, e.g., in a PERT network |
| Density (see Fig. 6.7) | $f(x) = \begin{cases} \dfrac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{B(\alpha_1,\alpha_2)} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ where $B(\alpha_1,\alpha_2)$ is the *beta function*, defined by $$B(z_1,z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\,dt$$ for any real numbers $z_1 > 0$ and $z_2 > 0$. Some properties of the beta function: $$B(z_1,z_2) = B(z_2,z_1), \qquad B(z_1,z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$$ |
| Distribution | No closed form, in general. If either $\alpha_1$ or $\alpha_2$ is a positive integer, a binomial expansion can be used to obtain $F(x)$, which will be a polynomial in $x$, and the powers of $x$ will be, in general, positive real numbers ranging from 0 through $\alpha_1 + \alpha_2 - 1$ |
| Parameters | Shape parameters $\alpha_1 > 0$ and $\alpha_2 > 0$ |
| Range | [0,1] |
| Mean | $\dfrac{\alpha_1}{\alpha_1+\alpha_2}$ |
| Variance | $\dfrac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2(\alpha_1+\alpha_2+1)}$ |
| Mode | $\begin{cases} \dfrac{\alpha_1-1}{\alpha_1+\alpha_2-2} & \text{if } \alpha_1 > 1, \alpha_2 > 1 \\ 0 \text{ and } 1 & \text{if } \alpha_1 < 1, \alpha_2 < 1 \\ 0 & \text{if } (\alpha_1 < 1, \alpha_2 \geq 1) \text{ or if } (\alpha_1 = 1, \alpha_2 > 1) \\ 1 & \text{if } (\alpha_1 \geq 1, \alpha_2 < 1) \text{ or if } (\alpha_1 > 1, \alpha_2 = 1) \\ \text{does not uniquely exist} & \text{if } \alpha_1 = \alpha_2 = 1 \end{cases}$ |
| MLE | The following two equations must be satisfied: $$\Psi(\hat{\alpha}_1) - \Psi(\hat{\alpha}_1 + \hat{\alpha}_2) = \ln G_1, \qquad \Psi(\hat{\alpha}_2) - \Psi(\hat{\alpha}_1 + \hat{\alpha}_2) = \ln G_2$$ where $\Psi$ is the digamma function, $G_1 = (\Pi_{i=1}^n X_i)^{1/n}$, and $G_2 = [\Pi_{i=1}^n (1 - X_i)]^{1/n}$ [see Gnanadesikan, Pinkham, and Hughes (1967)]; note that $G_1 + G_2 \leq 1$. These equations could be solved numerically [see Beckman and Tietjen (1978)], or approximations to $\hat{\alpha}_1$ and $\hat{\alpha}_2$ can be obtained from Table 6.20 (see App. 6A), which was computed for particular $(G_1,G_2)$ pairs by modifications of the methods in Beckman and Tietjen (1978) |
| Comments | 1. The U(0,1) and beta(1,1) distributions are the same<br>2. If $X_1$ and $X_2$ are independent random variables with $X_i \sim$ gamma$(\alpha_i,\beta)$, then $X_1/(X_1+X_2) \sim$ beta$(\alpha_1,\alpha_2)$<br>3. A beta random variable $X$ on [0,1] can be rescaled and relocated to obtain a beta random variable on $[a,b]$ of the same shape by the transformation $a + (b-a)X$ |

**TABLE 6.3** *(continued)*

| **Beta** | **beta($\alpha_1,\alpha_2$)** |
|---|---|
| | 4. $X \sim \text{beta}(\alpha_1,\alpha_2)$ if and only if $1 - X \sim \text{beta}(\alpha_2,\alpha_1)$ <br> 5. $X \sim \text{beta}(\alpha_1,\alpha_2)$ if and only if $Y = X/(1-X)$ has a Pearson type VI distribution with shape parameters $\alpha_1$, $\alpha_2$ and scale parameter 1, denoted PT6($\alpha_1,\alpha_2$,1) <br> 6. The beta(1,2) density is a left triangle, and the beta(2,1) density is a right triangle <br> 7. $\lim_{x\to 0} f(x) = \begin{cases} \infty & \text{if } \alpha_1 < 1 \\ \alpha_2 & \text{if } \alpha_1 = 1 \\ 0 & \text{if } \alpha_1 > 1 \end{cases}$, $\lim_{x\to 1} f(x) = \begin{cases} \infty & \text{if } \alpha_2 < 1 \\ \alpha_1 & \text{if } \alpha_2 = 1 \\ 0 & \text{if } \alpha_2 > 1 \end{cases}$ <br> 8. The density is symmetric about $x = \frac{1}{2}$ if and only if $\alpha_1 = \alpha_2$. Also, the mean and the mode are equal if and only if $\alpha_1 = \alpha_2$ |

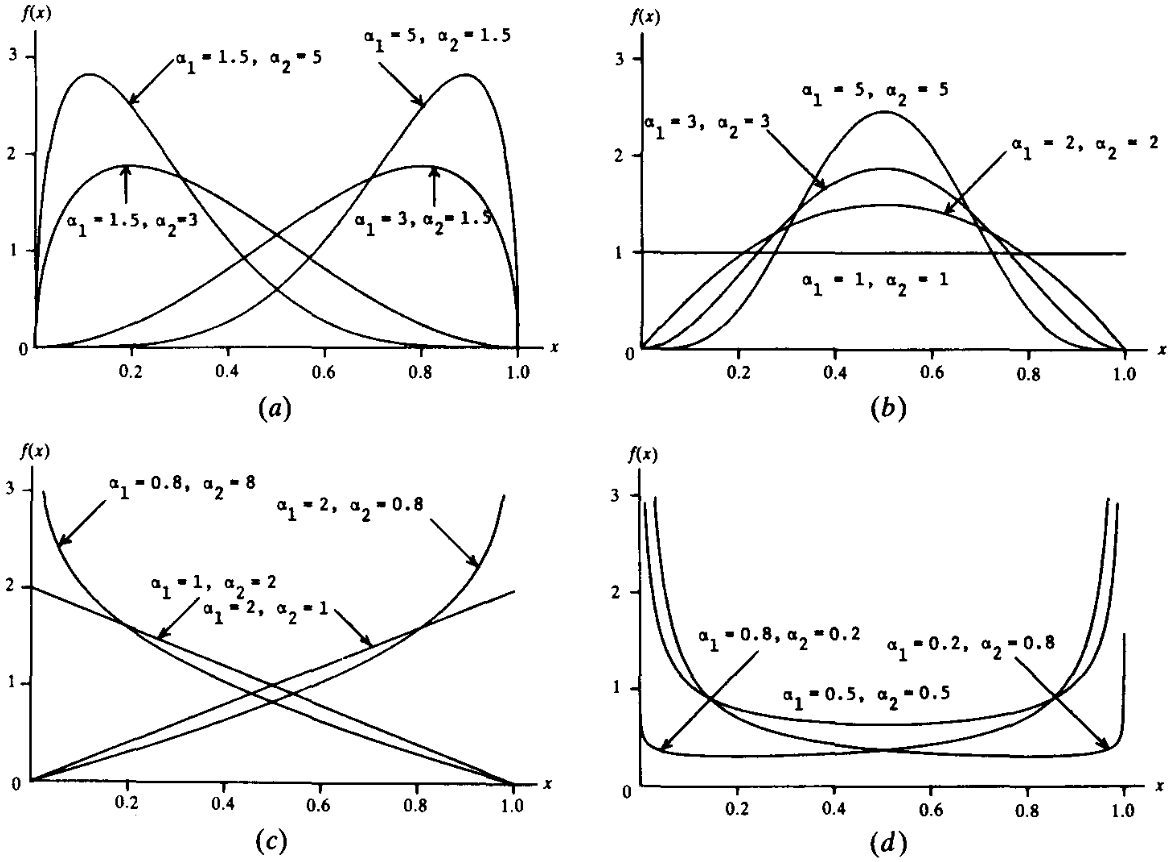


**FIGURE 6.7**
beta($\alpha_1,\alpha_2$) density functions.

| **Pearson type V** | **PT5($\alpha,\beta$)** |
|---|---|
| Possible applications | Time to perform some task (density takes on shapes similar to lognormal, but can have a larger "spike" close to $x = 0$) |
| Density (see Fig. 6.8) | $f(x) = \begin{cases} \dfrac{x^{-(\alpha+1)}e^{-\beta/x}}{\beta^{-\alpha}\Gamma(\alpha)} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$ |

**TABLE 6.3** *(continued)*

| **Pearson type V** | **PT5($\alpha,\beta$)** |
|---|---|
| Distribution | $F(x) = \begin{cases} 1 - F_G\left(\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$ where $F_G(x)$ is the distribution function of a gamma($\alpha,1/\beta$) random variable |
| Parameters | Shape parameter $\alpha > 0$, scale parameter $\beta > 0$ |
| Range | $[0,\infty)$ |
| Mean | $\dfrac{\beta}{\alpha - 1}$ for $\alpha > 1$ |
| Variance | $\dfrac{\beta^2}{(\alpha-1)^2(\alpha-2)}$ for $\alpha > 2$ |
| Mode | $\dfrac{\beta}{\alpha + 1}$ |
| MLE | If one has data $X_1, X_2, \ldots, X_n$, then fit a gamma($\alpha_G,\beta_G$) distribution to $1/X_1, 1/X_2, \ldots, 1/X_n$, resulting in the maximum-likelihood estimators $\hat{\alpha}_G$ and $\hat{\beta}_G$. Then the maximum-likelihood estimators for the PT5($\alpha,\beta$) are $\hat{\alpha} = \hat{\alpha}_G$ and $\hat{\beta} = 1/\hat{\beta}_G$ (see comment 1 below) |
| Comments | 1. $X \sim$ PT5($\alpha,\beta$) if and only if $Y = 1/X \sim$ gamma($\alpha,1/\beta$). Thus, the Pearson type V distribution is sometimes called the *inverted gamma distribution* 2. Note that the mean and variance exist only for certain values of the shape parameter |

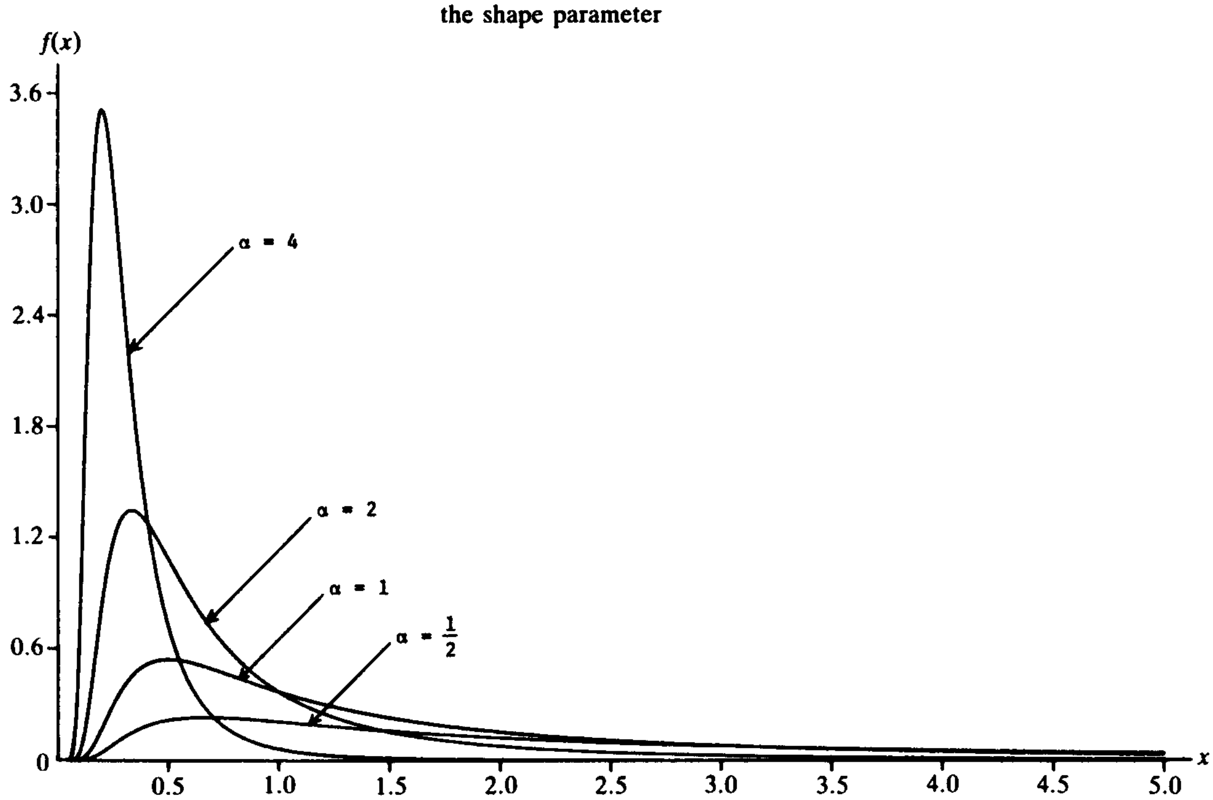


**FIGURE 6.8**
PT5($\alpha$,1) density functions.

**TABLE 6.3** *(continued)*

| **Pearson type VI** | **PT6($\alpha_1,\alpha_2,\beta$)** |
|---|---|
| Possible applications | Time to perform some task |
| Density (see Fig. 6.9) | $f(x) = \begin{cases} \dfrac{(x/\beta)^{\alpha_1 - 1}}{\beta B(\alpha_1,\alpha_2)[1 + (x/\beta)]^{\alpha_1+\alpha_2}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$ |
| Distribution | $F(x) = \begin{cases} F_B\left(\dfrac{x}{x+\beta}\right) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$ <br> where $F_B(x)$ is the distribution function of a beta($\alpha_1,\alpha_2$) random variable |
| Parameters | Shape parameters $\alpha_1 > 0$ and $\alpha_2 > 0$, scale parameter $\beta > 0$ |
| Range | $[0,\infty)$ |
| Mean | $\dfrac{\beta\alpha_1}{\alpha_2 - 1}$ for $\alpha_2 > 1$ |
| Variance | $\dfrac{\beta^2\alpha_1(\alpha_1 + \alpha_2 - 1)}{(\alpha_2 - 1)^2(\alpha_2 - 2)}$ for $\alpha_2 > 2$ |
| Mode | $\begin{cases} \dfrac{\beta(\alpha_1 - 1)}{\alpha_2 + 1} & \text{if } \alpha_1 \geq 1 \\ 0 & \text{otherwise} \end{cases}$ |
| MLE | If one has data $X_1, X_2, \ldots, X_n$ that are thought to be PT6($\alpha_1,\alpha_2$,1), then fit a beta($\alpha_1,\alpha_2$) distribution to $X_i/(1 + X_i)$ for $i = 1, 2, \ldots, n$, resulting in the maximum-likelihood estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$. Then the maximum-likelihood estimators for the PT6($\alpha_1,\alpha_2$,1) (note that $\beta = 1$) distribution are also $\hat{\alpha}_1$ and $\hat{\alpha}_2$ (see comment 1 below) |
| Comments | 1. $X \sim$ PT6($\alpha_1,\alpha_2$,1) if and only if $Y = X/(1 + X) \sim$ beta($\alpha_1,\alpha_2$) <br> 2. If $X_1$ and $X_2$ are independent random variables with $X_i \sim$ gamma($\alpha_i,\beta$), then $Y = X_1/X_2 \sim$ PT6($\alpha_1,\alpha_2,\beta$) (see Prob. 6.3) <br> 3. Note that the mean and variance exist only for certain values of the shape parameter $\alpha_2$ |

| **Triangular** | **triang($a,b,c$)** |
|---|---|
| Possible applications | Used as a rough model in the absence of data (see Sec. 6.9) |
| Density (see Fig. 6.10) | $f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)} & \text{if } a \leq x \leq c \\ \dfrac{2(b-x)}{(b-a)(b-c)} & \text{if } c < x \leq b \\ 0 & \text{otherwise} \end{cases}$ |
| Distribution | $F(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{(x-a)^2}{(b-a)(c-a)} & \text{if } a \leq x \leq c \\ 1 - \dfrac{(b-x)^2}{(b-a)(b-c)} & \text{if } c < x \leq b \\ 1 & \text{if } b < x \end{cases}$ |
| Parameters | $a$, $b$, and $c$ real numbers with $a < c < b$. $a$ is a location parameter, $b - a$ is a scale parameter, $c$ is a shape parameter |
| Range | $[a,b]$ |

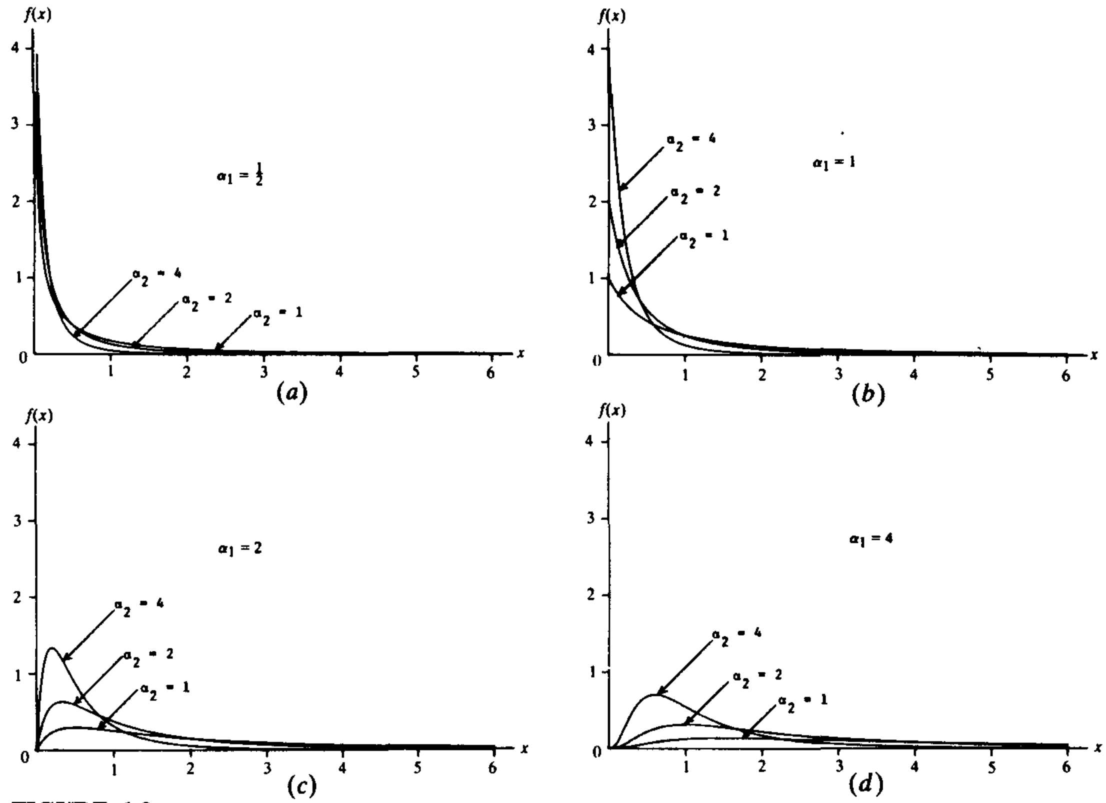


**FIGURE 6.9**
PT6($\alpha_1$,$\alpha_2$,1) density functions.

**TABLE 6.3** (*continued*)

| **Triangular** | **triang(*a*,*b*,*c*)** |
|---|---|
| Mean | $\frac{a+b+c}{3}$ |
| Variance | $\frac{a^2+b^2+c^2-ab-ac-bc}{18}$ |
| Mode | $c$ |
| MLE | Our use of the triangular distribution, as described in Sec. 6.9, is as a rough model when there are no data. Thus, MLEs are not relevant |
| Comment | The limiting cases as $c \to b$ and $c \to a$ are called the *right triangular* and *left triangular distributions*, respectively, and are discussed in Prob. 8.7. For $a = 0$ and $b = 1$, both the left and right triangular distributions are special cases of the beta distribution |

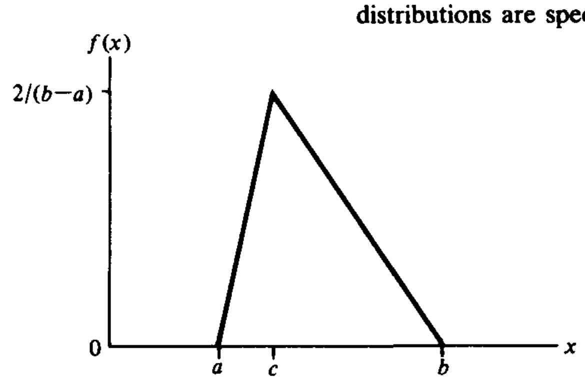


**FIGURE 6.10**
triang($a$,$b$,$c$) density functions.

Lawless (1982) for other applications]. Then the density function and distribution function (if it exists in simple closed form) are listed. Next is a short description of the parameters, including their possible values. The range indicates the interval where the associated random variable can take on values. Also listed are the mean (expected value), variance, and mode, i.e., the value at which the density function is maximized. MLE refers to the maximum-likelihood estimator(s) of the parameter(s), treated later in Sec. 6.5. General comments include relationships of the distribution under study to other distributions. Graphs are given of the density functions for each distribution. The notation following the name of each distribution is our abbreviation for that distribution, which includes the parameters. The symbol ~ is read "is distributed as."

Note that we have included the less familiar Pearson type V and Pearson type VI distributions, because we have found that these distributions often provide a better fit to data sets whose histograms are skewed to the right (see Fig. 6.19) than standard distributions such as gamma, Weibull, and lognormal.

### 6.2.3 Discrete Distributions

The descriptions of the six discrete distributions in Table 6.4 follow the same pattern as for the continuous distributions in Table 6.3.

**TABLE 6.4**
**Discrete distributions**

| **Bernoulli** | **Bernoulli($p$)** |
|---|---|
| Possible applications | Random occurrence with two possible outcomes; used to generate other discrete random variates, e.g., binomial, geometric, and negative binomial |
| Mass (see Fig. 6.11) | $p(x) = \begin{cases} 1-p & \text{if } x=0 \\ p & \text{if } x=1 \\ 0 & \text{otherwise} \end{cases}$ |
| Distribution | $F(x) = \begin{cases} 0 & \text{if } x<0 \\ 1-p & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \end{cases}$ |
| Parameter | $p \in (0,1)$ |
| Range | $\{0,1\}$ |
| Mean | $p$ |
| Variance | $p(1-p)$ |
| Mode | $\begin{cases} 0 & \text{if } p < \frac{1}{2} \\ 0 \text{ and } 1 & \text{if } p = \frac{1}{2} \\ 1 & \text{if } p > \frac{1}{2} \end{cases}$ |
| MLE | $\hat{p} = \bar{X}(n)$ |
| Comments | 1. A Bernoulli($p$) random variable $X$ can be thought of as the outcome of an experiment that either "fails" or "succeeds." If the probability of success is $p$, and we let $X = 0$ if the experiment |

**TABLE 6.4** *(continued)*

| **Bernoulli** | **Bernoulli($p$)** |
|---|---|
| | fails and $X = 1$ if it succeeds, then $X \sim$ Bernoulli($p$). Such an experiment, often called a *Bernoulli trial*, provides a convenient way of relating several other discrete distributions to the Bernoulli distribution<br>2. If $t$ is a positive integer and $X_1, X_2, \ldots, X_t$ are independent Bernoulli($p$) random variables, $X_1 + X_2 + \cdots + X_t$ has the binomial distribution with parameters $t$ and $p$. Thus, a binomial random variable can be thought of as the number of successes in a fixed number of independent Bernoulli trials<br>3. Suppose we begin making independent replications of a Bernoulli trial with probability $p$ of success on each trial. Then the number of failures *before* observing the first success has a geometric distribution with parameter $p$. For a positive integer $s$, the number of failures before observing the $s$th success has a negative binomial distribution with parameters $s$ and $p$<br>4. The Bernoulli($p$) distribution is a special case of the binomial distribution (with $t = 1$ and the same value for $p$) |

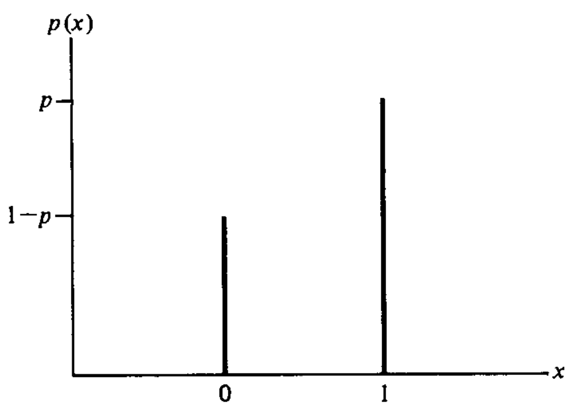


**FIGURE 6.11**
Bernoulli($p$) mass function ($p > 0.5$ here).

| **Discrete uniform** | **DU($i,j$)** |
|---|---|
| Possible applications | Random occurrence with several possible outcomes, each of which is equally likely; used as a "first" model for a quantity that is varying among the integers $i$ through $j$ but about which little else is known |
| Mass (see Fig. 6.12) | $p(x) = \begin{cases} \dfrac{1}{j-i+1} & \text{if } x \in \{i, i+1, \ldots, j\} \\ 0 & \text{otherwise} \end{cases}$ |
| Distribution | $F(x) = \begin{cases} 0 & \text{if } x < i \\ \dfrac{\lfloor x \rfloor - i + 1}{j-i+1} & \text{if } i \le x \le j \\ 1 & \text{if } j < x \end{cases}$<br>where $\lfloor x \rfloor$ denotes the largest integer $\le x$ |
| Parameters | $i$ and $j$ integers with $i \le j$; $i$ is a location parameter, $j - i$ is a scale parameter |
| Range | $\{i, i+1, \ldots, j\}$ |

TABLE 6.4 (*continued*)

| **Discrete uniform** | **DU($i,j$)** |
|---|---|
| Mean | $\dfrac{i+j}{2}$ |
| Variance | $\dfrac{(j-i+1)^2-1}{12}$ |
| Mode | Does not uniquely exist |
| MLE | $\hat{i} = \min_{1\le k\le n} X_k, \quad \hat{j} = \max_{1\le k\le n} X_k$ |
| Comment | The DU(0,1) and Bernoulli($\frac{1}{2}$) distributions are the same |

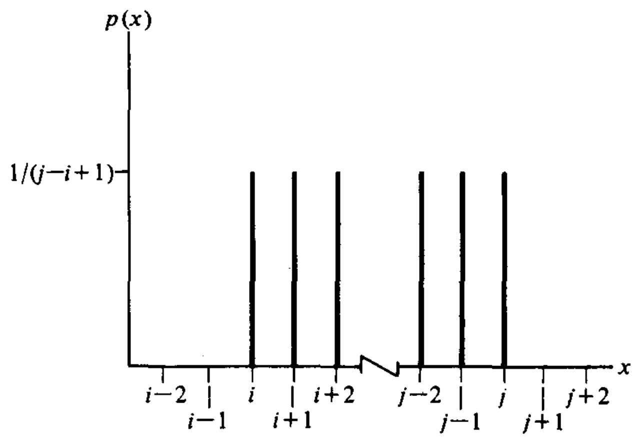


**FIGURE 6.12**
DU($i,j$) mass function.

| **Binomial** | **bin($t,p$)** |
|---|---|
| Possible applications | Number of successes in $t$ independent Bernoulli trials with probability $p$ of success on each trial; number of "defective" items in a batch of size $t$; number of items in a batch (e.g., a group of people) of random size; number of items demanded from an inventory |
| Mass (see Fig. 6.13) | $p(x) = \begin{cases} \binom{t}{x} p^x (1-p)^{t-x} & \text{if } x \in \{0, 1, \ldots, t\} \\ 0 & \text{otherwise} \end{cases}$ <br> where $\binom{t}{x}$ is the *binomial coefficient*, defined by <br> $\binom{t}{x} = \dfrac{t!}{x!(t-x)!}$ |
| Distribution | $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sum_{i=0}^{\lfloor x \rfloor} \binom{t}{i} p^i (1-p)^{t-i} & \text{if } 0 \le x \le t \\ 1 & \text{if } t < x \end{cases}$ |
| Parameters | $t$ a positive integer, $p \in (0,1)$ |
| Range | $\{0, 1, \ldots, t\}$ |
| Mean | $tp$ |
| Variance | $tp(1-p)$ |
| Mode | $\begin{cases} p(t+1)-1 \text{ and } p(t+1) & \text{if } p(t+1) \text{ is an integer} \\ \lfloor p(t+1) \rfloor & \text{otherwise} \end{cases}$ |
| MLE | If $t$ is known, then $\hat{p} = \bar{X}(n)/t$. If both $t$ and $p$ are unknown, then $\hat{t}$ and $\hat{p}$ exist if and only if $\bar{X}(n) > (n-1)S^2(n)/n = V(n)$. Then the |

**TABLE 6.4** (*continued*)

| **Binomial** | **bin($t, p$)** |
|---|---|

following approach could be taken. Let $M = \max_{1 \le i \le n} X_i$, and for $k = 0, 1, \ldots, M$, let $f_k$ be the number of $X_i$'s $\ge k$. Then it can be shown that $\hat{t}$ and $\hat{p}$ are the values for $t$ and $p$ that maximize the function

$$g(t, p) = \sum_{k=1}^{M} f_k \ln(t - k + 1) + nt \ln(1 - p) + n\bar{X}(n) \ln \frac{p}{1 - p}$$

subject to the constraints that $t \in \{M, M + 1, \ldots\}$ and $0 < p < 1$. It is easy to see that for a fixed value of $t$, say $t_0$, the value of $p$ that maximizes $g(t_0, p)$ is $\bar{X}(n)/t_0$, so $\hat{t}$ and $\hat{p}$ are the values of $t$ and $\bar{X}(n)/t$ that lead to the largest value of $g[t, \bar{X}(n)/t]$ for $t \in \{M, M + 1, \ldots, M'\}$, where $M'$ is given by [see DeRiggi (1983)]

$$M' = \left\lfloor \frac{\bar{X}(n)(M - 1)}{1 - [V(n)/\bar{X}(n)]} \right\rfloor$$

Note also that $g[t, \bar{X}(n)/t]$ is a unimodal function of $t$

**Comments**

1. If $Y_1, Y_2, \ldots, Y_t$ are independent Bernoulli($p$) random variables, then $Y_1 + Y_2 + \cdots + Y_t \sim \text{bin}(t, p)$
2. If $X_1, X_2, \ldots, X_m$ are independent random variables and $X_i \sim \text{bin}(t_i, p)$, then $X_1 + X_2 + \cdots + X_m \sim \text{bin}(t_1 + t_2 + \cdots + t_m, p)$
3. The bin($t, p$) distribution is symmetric if and only if $p = \frac{1}{2}$
4. $X \sim \text{bin}(t, p)$ if and only if $t - X \sim \text{bin}(t, 1 - p)$
5. The bin($1, p$) and Bernoulli($p$) distributions are the same

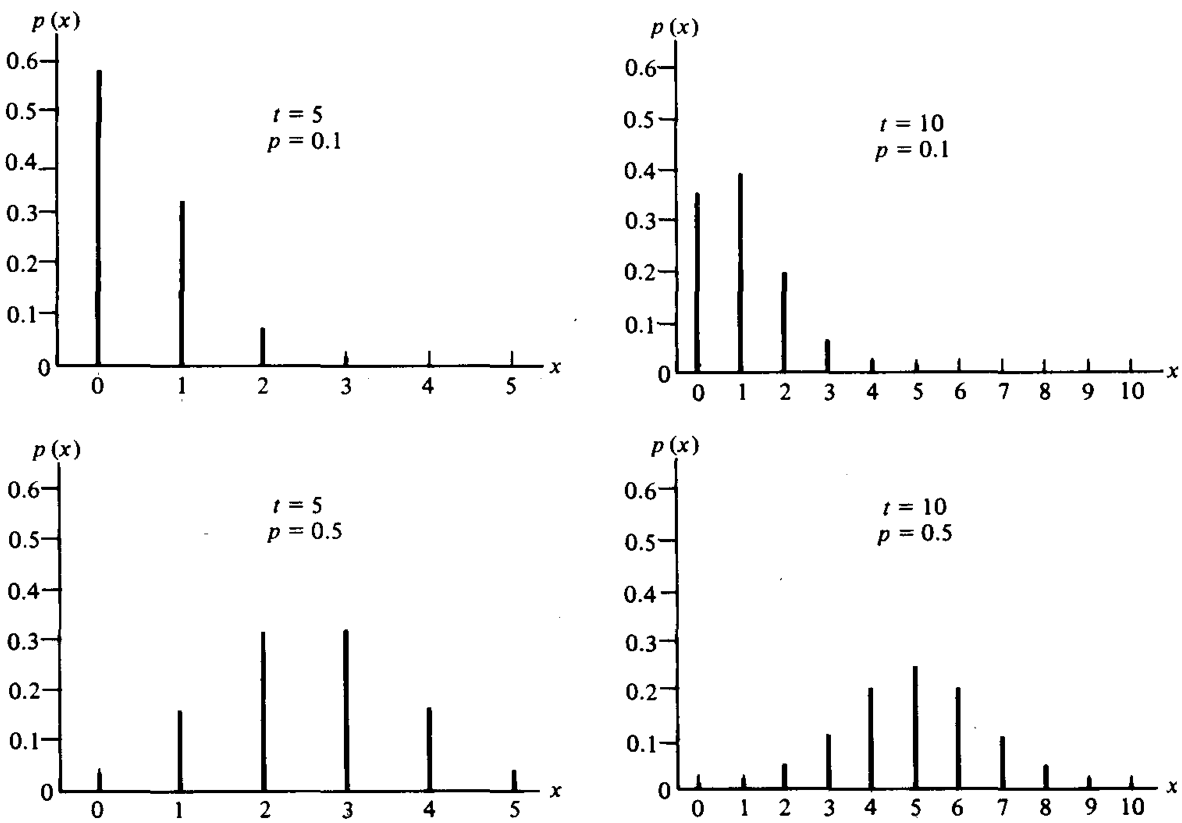


**FIGURE 6.13**
bin($t, p$) mass functions.

**TABLE 6.4** *(continued)*

| **Geometric** | **geom($p$)** |
|---|---|
| Possible applications | Number of failures before the first success in a sequence of independent Bernoulli trials with probability $p$ of success on each trial; number of items inspected before encountering the first defective item; number of items in a batch of random size; number of items demanded from an inventory |
| Mass (see Fig. 6.14) | $p(x) = \begin{cases} p(1-p)^x & \text{if } x \in \{0, 1, \ldots\} \\ 0 & \text{otherwise} \end{cases}$ |
| Distribution | $F(x) = \begin{cases} 1-(1-p)^{\lfloor x \rfloor + 1} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$ |
| Parameter | $p \in (0,1)$ |
| Range | $\{0, 1, \ldots\}$ |
| Mean | $\dfrac{1-p}{p}$ |
| Variance | $\dfrac{1-p}{p^2}$ |
| Mode | 0 |
| MLE | $\hat{p} = \dfrac{1}{\bar{X}(n)+1}$ |
| Comments | 1. If $Y_1, Y_2, \ldots$ is a sequence of independent Bernoulli($p$) random variables and $X = \min\{i: Y_i = 1\} - 1$, then $X \sim \text{geom}(p)$.<br>2. If $X_1, X_2, \ldots, X_s$ are independent geom($p$) random variables, then $X_1 + X_2 + \cdots + X_s$ has a negative binomial distribution with parameters $s$ and $p$<br>3. The geometric distribution is the discrete analog of the exponential distribution, in the sense that it is the only discrete distribution with the memoryless property (see Prob. 4.27)<br>4. The geom($p$) distribution is a special case of the negative binomial distribution (with $s = 1$ and the same value for $p$) |

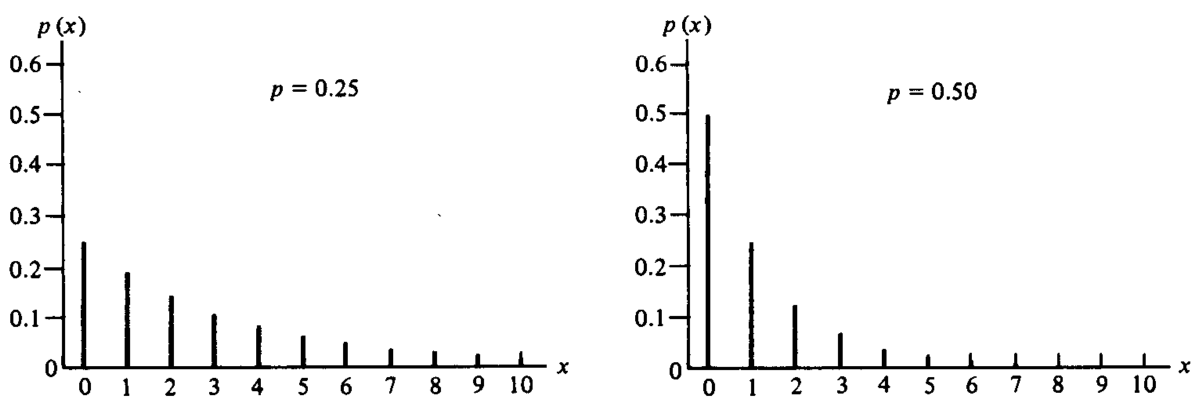


**FIGURE 6.14**
geom($p$) mass functions.

**TABLE 6.4** *(continued)*

| Negative binomial | negbin($s,p$) |
|---|---|
| Possible applications | Number of failures before the $s$th success in a sequence of independent Bernoulli trials with probability $p$ of success on each trial; number of good items inspected before encountering the $s$th defective item; number of items in a batch of random size; number of items demanded from an inventory |
| Mass (see Fig. 6.15) | $p(x)=\begin{cases}\binom{s+x-1}{x}p^s(1-p)^x & \text{if } x\in\{0,1,\ldots\}\\ 0 & \text{otherwise}\end{cases}$ |
| Distribution | $F(x)=\begin{cases}\sum_{i=0}^{\lfloor x\rfloor}\binom{s+i-1}{i}p^s(1-p)^i & \text{if } x\geq 0\\ 0 & \text{otherwise}\end{cases}$ |
| Parameters | $s$ a positive integer, $p\in(0,1)$ |
| Range | $\{0,1,\ldots\}$ |
| Mean | $\frac{s(1-p)}{p}$ |
| Variance | $\frac{s(1-p)}{p^2}$ |
| Mode | Let $y=[s(1-p)-1]/p$; then $\text{Mode}=\begin{cases}y \text{ and } y+1 & \text{if } y \text{ is an integer}\\ \lfloor y\rfloor+1 & \text{otherwise}\end{cases}$ |
| MLE | If $s$ is known, then $\hat{p}=s/[\bar{X}(n)+s]$. If both $s$ and $p$ are unknown, then $\hat{s}$ and $\hat{p}$ exist if and only if $V(n)=(n-1)S^2(n)/n>\bar{X}(n)$. Let $M=\max_{1\leq i\leq n} X_i$, and for $k=0,1,\ldots,M$, let $f_k$ be the number of $X_i$'s $\geq k$. Then we can show that $\hat{s}$ and $\hat{p}$ are the values for $s$ and $p$ that maximize the function $h(s,p)=\sum_{k=1}^{M} f_k \ln(s+k-1)+ns\ln p+n\bar{X}(n)\ln(1-p)$ subject to the constraints that $s\in\{1,2,\ldots\}$ and $0<p<1$. For a fixed value of $s$, say $s_0$, the value of $p$ that maximizes $h(s_0,p)$ is $s_0/[\bar{X}(n)+s_0]$, so that we could examine $h(1,1/[\bar{X}(n)+1])$, $h(2,2/[\bar{X}(n)+2]),\ldots$. Then $\hat{s}$ and $\hat{p}$ are chosen to be the values of $s$ and $s/[\bar{X}(n)+s]$ that lead to the biggest observed value of $h(s,s/[\bar{X}(n)+s])$. However, since $h(s,s/[\bar{X}(n)+s])$ is a unimodal function of $s$ [see Levin and Reeds (1977)], it is clear when to terminate the search |
| Comments | 1. If $Y_1,Y_2,\ldots,Y_s$ are independent geom($p$) random variables, then $Y_1+Y_2+\cdots+Y_s\sim$ negbin($s,p$)<br>2. If $Y_1,Y_2,\ldots$ is a sequence of independent Bernoulli($p$) random variables and $X=\min\{i: \Sigma_{j=1}^{i} Y_j=s\}-s$, then $X\sim$ negbin($s,p$)<br>3. If $X_1,X_2,\ldots,X_m$ are independent random variables and $X_i\sim$ negbin($s_i,p$), then $X_1+X_2+\cdots+X_m\sim$ negbin($s_1+s_2+\cdots+s_m,p$)<br>4. The negbin($1,p$) and geom($p$) distributions are the same |

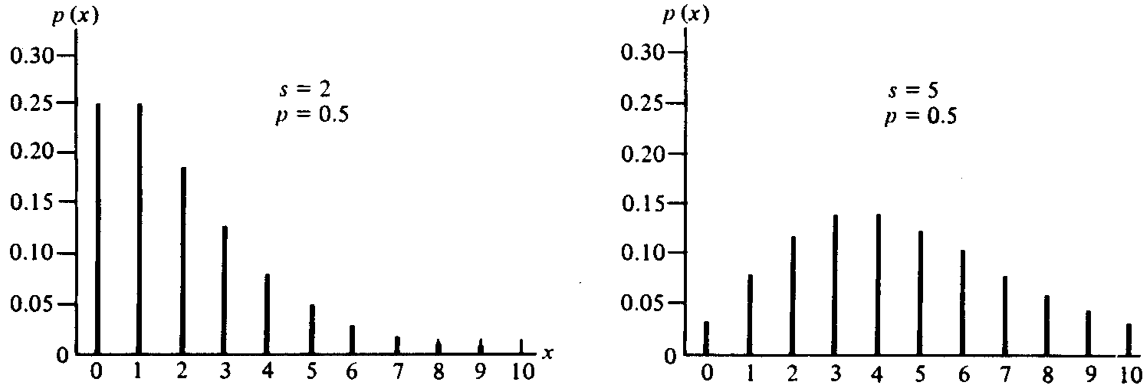


**FIGURE 6.15**
negbin($s,p$) mass functions.

**TABLE 6.4** *(continued)*

| **Poisson** | **Poisson($\lambda$)** |
|---|---|
| Possible applications | Number of events that occur in an interval of time when the events are occuring at a constant rate (see Sec. 6.10); number of items in a batch of random size; number of items demanded from an inventory |
| Mass (see Fig. 6.16) | $p(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & \text{if } x \in \{0, 1, \ldots\} \\ 0 & \text{otherwise} \end{cases}$ |
| Distribution | $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-\lambda} \sum_{i=0}^{\lfloor x \rfloor} \dfrac{\lambda^i}{i!} & \text{if } 0 \le x \end{cases}$ |
| Parameter | $\lambda > 0$ |
| Range | $\{0, 1, \ldots\}$ |
| Mean | $\lambda$ |
| Variance | $\lambda$ |
| Mode | $\begin{cases} \lambda - 1 \text{ and } \lambda & \text{if } \lambda \text{ is an integer} \\ \lfloor \lambda \rfloor & \text{otherwise} \end{cases}$ |
| MLE | $\hat{\lambda} = \bar{X}(n)$ |
| Comments | 1. Let $Y_1, Y_2, \ldots$ be a sequence of nonnegative IID random variables, and let $X = \max\{i: \Sigma_{j=1}^{i} Y_j \le 1\}$. Then the distribution of the $Y_i$'s is expo($1/\lambda$) if and only if $X \sim$ Poisson($\lambda$). Also, if $X' = \max\{i: \Sigma_{j=1}^{i} Y_j \le \lambda\}$, then the $Y_i$'s are expo(1) if and only if $X' \sim$ Poisson($\lambda$) (see also Sec. 6.10)<br>2. If $X_1, X_2, \ldots, X_m$ are independent random variables and $X_i \sim$ Poisson($\lambda_i$), then $X_1 + X_2 + \cdots + X_m \sim$ Poisson($\lambda_1 + \lambda_2 + \cdots + \lambda_m$) |

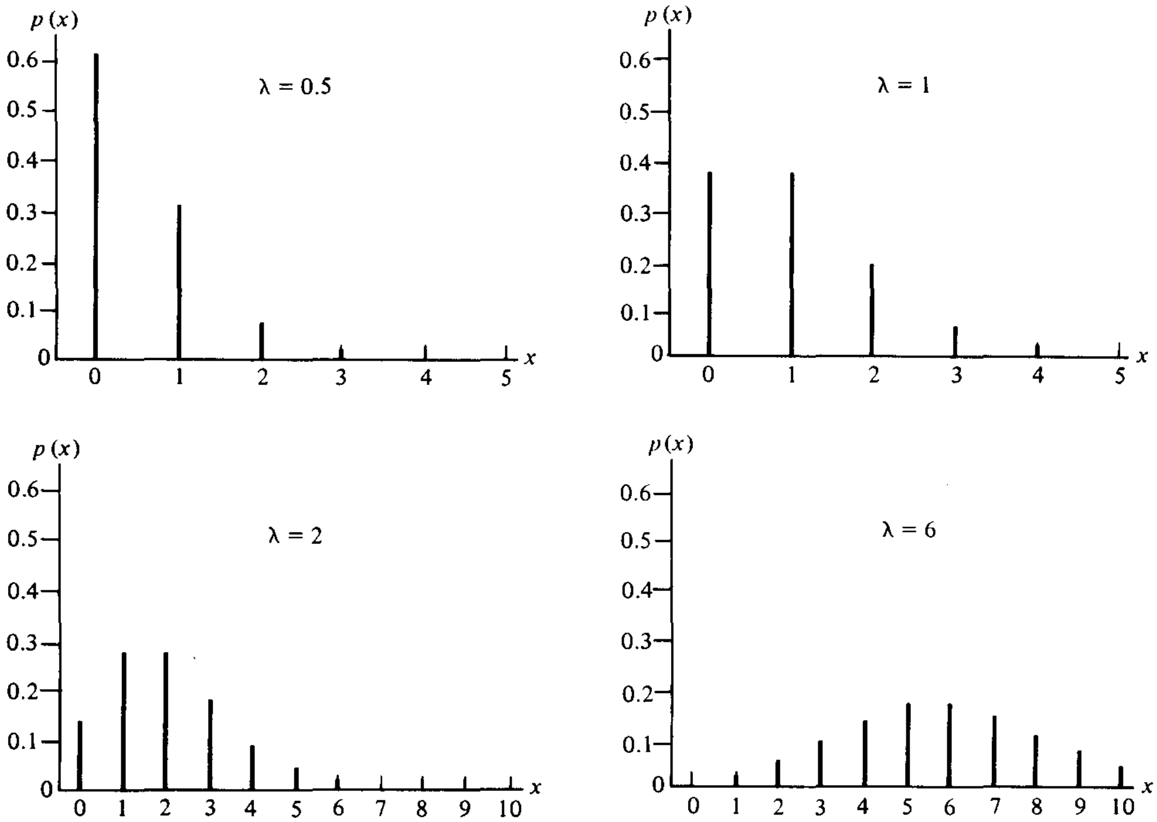


**FIGURE 6.16**
Poisson(λ) mass functions.

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Tomado de las págs.330-350 de *"Simulation Modeling & Análisis"* (2º edición)
Averill M. Law y W. David Kelton , McGraw-Hill, 1991